Chapter 6 Maxwell's Equations for Time-Varying Fields

If the charge or current sources vary with time, electric & magnetic filds become interconnected and electromagnetic waves are produced. EM waves can travel in free space and in materials.

Examples of EM waves are: light, X-ray, infrared waves, gamma rays, radio frequency (RF) waves.

Time-independent fields → Electrostatic & Magnetostatic, $\frac{\partial}{\partial t} = 0$

Time varying fields → Electromagnetics or Dynamic fields, $\frac{\partial}{\partial t} \neq 0$

We have now use the full version of Maxwell's equations:

① Gauss's law $\quad \vec{\nabla}\cdot\vec{D} = \rho_v$ (Same as static fields) $\rightarrow \oint_S \vec{D}\cdot d\vec{s} = Q$

② Faraday's law $\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$

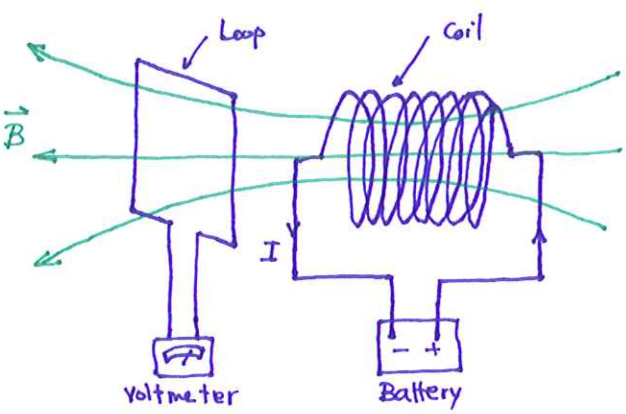
③ No magnetic charges $\quad \vec{\nabla}\cdot\vec{B} = 0$ (same as static) $\longrightarrow \oint_S \vec{B}\cdot d\vec{s} = 0$

④ Ampère's law $\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow \oint_C \vec{H}\cdot d\vec{l} = \int_S (\vec{J} + \frac{\partial \vec{D}}{\partial t})\cdot d\vec{s}$

Faraday's Law

We learned that a wire carrying current can generate magnetic field. Faraday thought that this should also work in reverse meaning that: a magnetic field should produce a current in a wire. He made many experiments to prove his hypothesis. Henry also carried similar experiments. (Faraday in London and Henry in Albany, NY). The concluded that:

magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time.



Connecting and disconnecting the battery induces a varrying current hence linking Φ in the Coil that produces a current (or voltage) in the loop.

Linking magnetic flux: $\Phi = \int_S \vec{B} \cdot \vec{ds}$ (Wb)

Produced voltag in a closed conducting loop of N turns:

$$\boxed{V_{emf} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \vec{B} \cdot \vec{ds}}$$ Faraday's Law

total derivative. So either changes in B or S can produce voltage (V_{emf})

V_{emf}: electrmotive force

This process is called electromagnetic induction.

An emf can be generated under any of the following conditions:

Case ① A time-varying $\vec{B}$ linking a stationary loop → transformer emf, V_{emf}^{tr}

Case ② A moving loop with a time varying area in a static field $\vec{B}$ → motional emf, V_{emf}^{m}
(area must vary relative to the normal component of B)

Case ③ A moving loop in a time-varying field $\vec{B}$

The total emf is given by: $V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$

If the loop is stationary: $V_{emf}^{m} = 0$

If $\vec{B}$ is static: $V_{emf}^{tr} = 0$

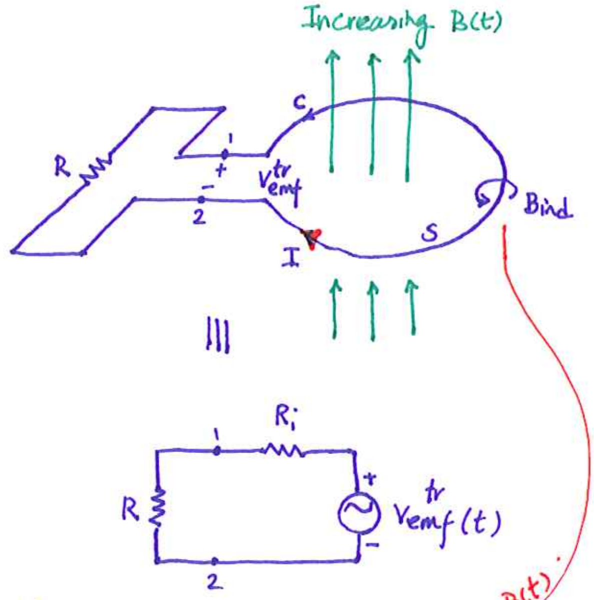
Case 1: Stationary Loop in a Time-Varying Magnetic Field

We'll have the transformer emf in this case:

$$V_{emf} = -N \frac{d}{dt} \int \vec{B} \cdot \vec{ds} \quad ; \quad \frac{d}{dt} \int (B \cdot ds) = \int \frac{dB}{dt} \cdot ds + \underbrace{\int B \cdot d\left(\frac{ds}{dt}\right)}_{=0 \text{ since } s \text{ is not changing}} = \int \frac{\partial B}{\partial t} \cdot ds$$

$$\Rightarrow \quad \boxed{V_{emf}^{tr} = -N \int_S \frac{\partial \vec{B}}{\partial t} \cdot ds}$$

As an example consider a loop of conductor in a varying magnetic field as shown here.



A V_{emf}^{tr} is generated across points 1 and 2 in the figure that produces a current in the load R. If the resistance of the loop itself is R_i, we have: $I = \dfrac{V_{emf}^{tr}}{R+R_i}$

The polarity of V_{emf}^{tr} is governed by Lenz's Law:

▷ The voltage hence the resulting current in the loop is such that it opposes the change of magnetic flux $\phi(t)$ that produced it. ◁ $\vec{B}_{ind}$ opposes increase in B(t).

Basically the current I induces a field $\vec{B}_{ind}$. If $\vec{B}(t)$ is increasing $\vec{I}$ is in a direction so that $\vec{B}_{ind}$ is in opposite direction to $\vec{B}(t)$. If B(t) is decreasing $\vec{I}$ changes to a direction so that $\vec{B}_{ind}$ is in same direction as $\vec{B}(t)$ so to help it.

The V_{emf}^{tr} is related to the electric field induced in the ring as:

$$V_{emf}^{tr} = \oint_c \vec{E}\cdot d\vec{l} \qquad V_{emf}^{tr} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

To get the Maxwell's 2^{nd} equ. we have: $\oint_c \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$ (Faraday's law)

$$\downarrow$$

$$\int_s (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

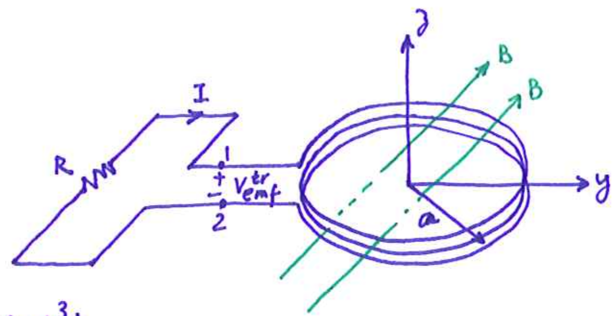
$$\Rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ (The differential form of Faraday's law)

Example

An inductor with N turns is making a circular loop of radius a. The inductor loop is in x-y plane and is connected to a resistor R as shown. In the presence of magnetic field $\vec{B} = B_0(\hat{y}2+\hat{z}3)\sin\omega t$, find: a) magnetic flux linking a single turn of inductor b) V_{emf}^{tr} if $N=10$, $B_0=0.2T$, $a=10cm$, $\omega=10^3$ rad/s c) the polarity of V_{emf}^{tr} at $t=0$ and d) induced current in the circuit for $R=1\,k\Omega$

(a) $\Phi = \int_S \vec{B} \cdot \vec{ds}$

$$= \int_S \overbrace{[B_0 (\hat{y}2 + \hat{z}3) \sin\omega t]}^{\vec{B}} \cdot \overbrace{\hat{z}\, ds}^{\vec{ds}} = 3\pi a^2 B_0 \sin\omega t$$



(b) $V_{emf}^{tr} = -N \frac{d\Phi}{dt} = -10 \frac{d}{dt}(3\pi a^2 B_0 \sin\omega t) = -188.5 \cos 10^3 t$

(c) Since $\frac{d\Phi}{dt} > 0$ at $t=0$ (and $V_{emf}^{tr} = -188.5\,V$) the flux is increasing so current I is in the direction shown in the picture to oppose the field based on Lenz's law $\Rightarrow V_1 - V_2 < 0$

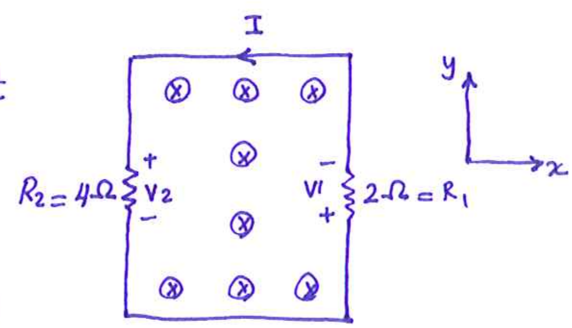
$$V_{emf}^{tr} = V_1 - V_2 = -188.5 \ (V)$$

(d) The current I is given by: $I = \frac{V_2 - V_1}{R} = \frac{188.5}{10^3} \cos 10^3 t = 0.19 \cos 10^3 t$

Example A loop of conductor is located in the xy plane. It's area is $4m^2$ and $\vec{B} = -\hat{z}\,0.3t$ (T). The internal resistance of the loop is ignorable. Determine the voltages V_1 and V_2 across the 2Ω and 4Ω resistors.

Solution: $\Phi = \int_S \vec{B} \cdot \vec{ds} = \int_S (-\hat{z}\,0.3t) \cdot \hat{z}\, ds = -0.3t \times 4 = -1.2t$

$V_{emf}^{tr} = -\frac{d\Phi}{dt} = 1.2 \ (V)$



Since B is increasing with time in $-\hat{z}$ direction,

Curren I will be Counterclockwise to oppose this field.

$$I = \frac{V_{emf}^{tr}}{R_1 + R_2} = \frac{1.2}{2+4} = 0.2\,A \rightarrow V_1 = R_1 I = 0.2 \times 2 = 0.4\,V$$

$$V_2 = R_2 I = 4 \times 0.2 = 0.8\,V$$

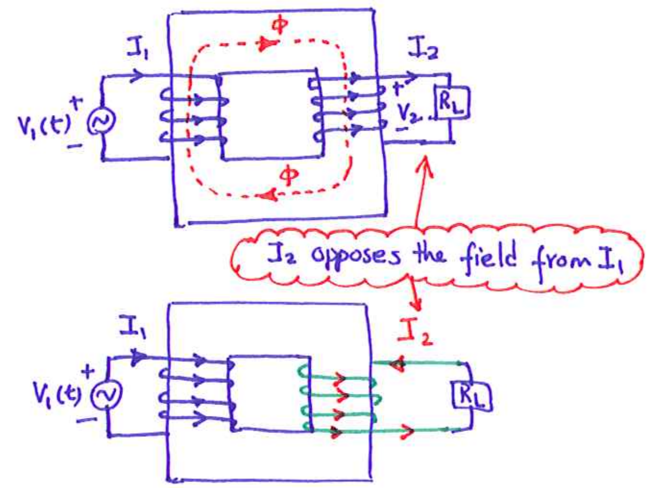
The Ideal Transformer

In a transformer there are two coils usually around a single magnetic core. The primary coil is connected to an a-c voltage source $V_1(t)$ and the secondary coil to a load resistor R_L. In an ideal transformer the core has infinite permeability: ($\mu = \infty$), and the magnetic flux is connected through the core.

The direction of I_1 and I_2 are such that their flux opposes each other (Lenz's law).

From Faraday's law:

$$V_1 = -N_1 \frac{d\phi}{dt}$$



Similarly for the secondary side:

$$V_2 = -N_2 \frac{d\phi}{dt}$$

$$\Rightarrow -\frac{d\phi}{dt} = \frac{V_1}{N_1} = \frac{V_2}{N_2} \rightarrow \boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}}$$

In an ideal transformer there is no power loss, so the input power $V_1 I_1$ must be equal to the output power $V_2 I_2$:

$$\begin{cases} P_1 = V_1 I_1 \\ P_2 = V_2 I_2 \end{cases} \xrightarrow{\text{lossless}} P_1 = P_2 \rightarrow V_1 I_1 = V_2 I_2 \rightarrow \frac{V_1}{V_2} = \frac{I_2}{I_1} \rightarrow$$

$$\boxed{\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2}}$$

If we look at the input impedance in the primary coil in the figure:

$$R_{in} = \frac{V_1}{I_1}$$

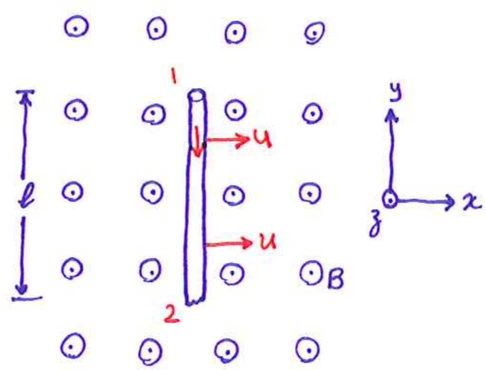
$$\left.\begin{array}{l} \text{But } V_1 = V_2 \dfrac{N_1}{N_2} \text{ and } I_1 = \dfrac{N_2}{N_1} I_2 \end{array}\right\} R_{in} = \frac{V_2}{I_2} \left(\frac{N_1}{N_2}\right)^2 = R_L \left(\frac{N_1}{N_2}\right)^2$$

If the load has impedace Z_L, we can extend the above relation to:

$$\boxed{Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L}$$

Case 2: Moving Conductor in a Static Magnetic Field

Now consider a moving conductor in a magnetic fied $\vec{B} = \hat{z} B_0$. The conductor has free electrons which are being forced by magnetic field as they are moving with the conductor. Considering the figure, the force F_m that moves the electrons is:



$$\vec{F}_m = q(\vec{u} \times \vec{B})$$

This force is equal to the electric force from an electric field:

$$\vec{E}_m = \frac{\vec{F}_m}{q} = \vec{u} \times \vec{B}$$

$\vec{E}_m$ is called motional electric field as shown in the picture. The force pushes the electrons to one end of electrode inducing a voltage difference across the electrode:

$$V_{emf}^{m} = V_{12} = \int_2^1 \vec{E}_m \cdot \vec{dl} = \int_2^1 (\vec{u} \times \vec{B}) \cdot \vec{dl} = \int_2^1 (\hat{x} u \times \hat{z} B_0) \cdot (\hat{y}\, dl)$$

$$= \int_2^1 (-u B_0 \hat{y}) \cdot (\hat{y}\, dl) = -u B_0 l$$

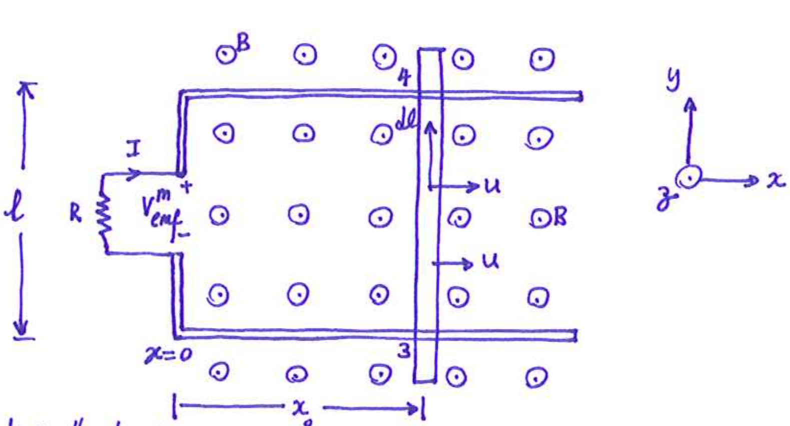
In general we can write for a segment of a closed circuit:

$$\boxed{V_{emf}^{m} = \oint_c (\vec{u} \times \vec{B}) \cdot dl}$$

Note that only those segments of the circuit that cross the magnetic field lines contribute to V_{emf}^{m}.

Example Sliding Bar

In the figure a bar of length l is moving in a magnetic field $\vec{B} = \hat{z} B_0 x$. The bar closes a circuit as show in the figure.



If the resistanc of the circuit is very small $R_i \ll R$, find the V_{emf} and current in the Load.

Solution:

$$V^m_{emf} = V_{12} = V_{43} = \int_3^4 (\vec{u} \times \vec{B}) \cdot \vec{dl} = \int_3^4 (\hat{x} u \times \hat{z} B_0 x_0) \cdot \hat{y}\, dl = -u B_0 x_0 l$$

Since $x_0 = ut \rightarrow V^m_{emf} = -B_0 u^2 l t$

Since B is static, $V^{tr}_{emf} = 0$ and $V_{emf} = V^m_{emf}$ only. $\rightarrow I = \frac{-V_{emf}}{R} = \frac{B_0 u^2 l t}{R}$

we can derive same result using Faraday's law:

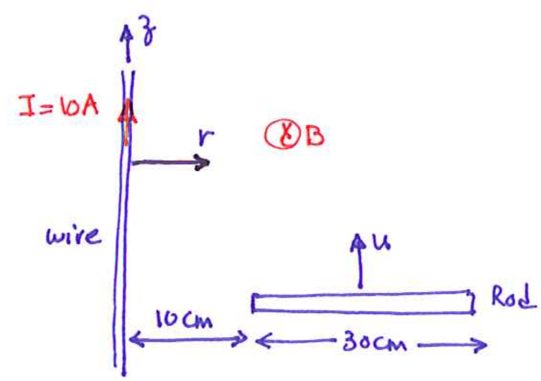
$$\Phi = \int_s B \cdot ds = \int_s (\hat{z} B_0 x) \cdot (\hat{z}\, dx\, dy) = B_0 \int_0^l dy \int_0^{x_0} dx\, x = B_0 l \frac{x_0^2}{2}$$

Since $x_0 = ut \rightarrow \Phi = \frac{B_0 l}{2} u^2 t^2$

$$V_{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{1}{2} B_0 l u^2 t^2\right) = -B_0 u^2 l t$$

Example Moving Rod Next to a wire.

As shown in the picture, a wire with current $I = 10A$ is in free space. A 30 cm long rod moves at constant velocity $u = \hat{z} 5 \frac{m}{s}$. Find V_{12}.

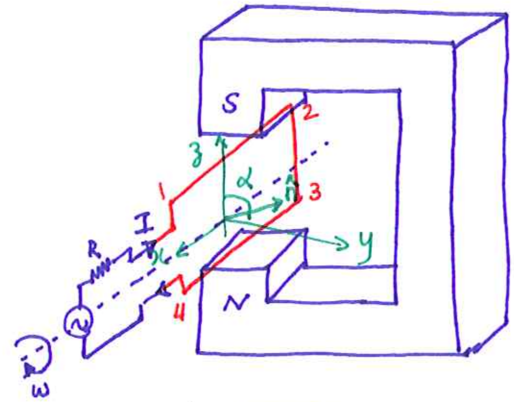


$$\vec{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \rightarrow V_{12} = \int_{40cm}^{10cm} (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

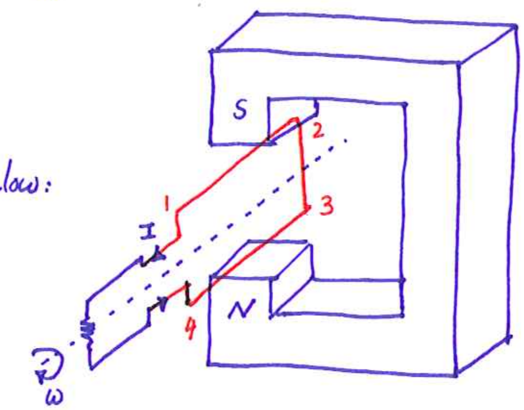
$$= \int_{40}^{10} \left(\hat{z} 5 \times \hat{\phi} \frac{\mu_0 I}{2\pi r}\right) \cdot \hat{r}\, dr = -\frac{5\mu_0 I}{2\pi} \int_{40}^{10} \frac{dr}{r} = -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \ln \frac{10}{40} = 13.9\ \mu V$$

The Electromagnetic Generator

An electromagnetic generator is inverse of an electromagnetic motor. It consists of a magnet and conductor loops that rotate in the field of the magnet as shown. If the magnet field is $\vec{B}=\hat{z}B_0$ and the axis of the rotation is x axis and segments of 1-2 and 3-4 have length l each, and the other segments w, we can calculate V_{emf}^{m} as below: segments 1-4 and 2-3 don't cross the field hence generate no voltage. only segments 1-2 and 3-4 generate voltage. If



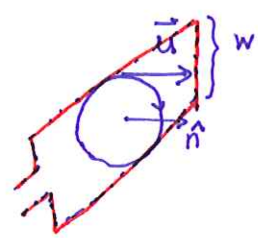
AC Motor



AC Generator

the loop rotates with angular velocity ω, the velocity u is:

$$U = R\omega \implies \vec{u} = \hat{n}\,\omega\,\frac{w}{2} \qquad \hat{n} \text{ is normal to the loop surface}$$



$$V_{emf}^{m} = V_{14} = \int_2^1 (\vec{u}\times\vec{B})\cdot d\vec{l} + \int_4^3 (\vec{u}\times\vec{B})\cdot d\vec{l}$$

$$= \int_{-l/2}^{l/2} \left[\left(\hat{n}\omega\frac{w}{2}\right)\times\hat{z}B_0\right]\cdot\hat{x}\,dx + \int_{-l/2}^{l/2} \left[\left(-\hat{n}\omega\frac{w}{2}\right)\times\hat{z}B_0\right]\cdot\hat{x}\,dx \qquad \hat{n}\times\hat{z} = \hat{x}\sin\alpha$$

$$= \overbrace{wl}^{A}\,\omega B_0\sin\alpha = A\omega B_0\sin\alpha$$

$$\alpha = \omega t + C_0 \rightarrow V_{emf}^{m} = A\omega B_0\sin(\omega t + C_0)$$

We may get same result using Faraday's law: $\Phi = \int_S \vec{B}\cdot d\vec{s} = \int_S \hat{z}B_0\cdot\hat{n}\,ds$

$$= B_0 A\cos\alpha = B_0 A\cos(\omega t + C_0) \rightarrow V_{emf} = -\frac{d\Phi}{dt} = A\omega B_0\sin(\omega t + C_0)$$

Case 3: Moving Conductor in a Time-Varying Magnetic Field

For a general case consider a single-turn conducting loop moving in a time-varying magnetic fied. The induced emf is now the sum of a transformer component and a motional component:

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

$$= \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$$

we can also equally calculate V_{emf} from Faraday's laws as usual:

$$\boxed{V_{emf} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}}$$

Example Electromagnetic Generator:

Find the induced voltage when the rotating loop of the EM generator of previous section is in a magnetic field $B = \hat{z}\, B_0 \cos\omega t$. Assume $\alpha = 0$ at $t = 0$.

Solution: we had $\Phi = B_0 A \cos\omega t$ Now B_0 is replaced with $B_0 \cos\omega t \rightarrow \Phi = B_0 A \cos^2\omega t$

$$V_{emf} = -\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial t}(B_0 A \cos^2\omega t) = 2 B_0 A \omega \cos\omega t \sin\omega t = B_0 A \omega \sin 2\omega t$$